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Supergravity with a Noninvertible Vierbein

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Abstract

We show that there is no off-shell Palatini formulation of minimal supergravity. Nonetheless, we have been able to generalize the multiplet and Lagrangians of this theory to the case of vanishing determinant of the vierbein. Unfortunately, the requirement of regularity does not single out a unique action.

1 Introduction

In [1] it was shown that altering the field content of general relativity to allow for noninvertible metrics gives the Einstein-Hilbert action as the unique action describing pure gravity in four spacetime dimensions. It was also shown, however, that this uniqueness is lost when coupling matter to gravity and is only regained in spacetimes with dimensions greater than or equal to nine.

It was hoped that an additional symmetry, namely supersymmetry, could help to establish this uniqueness in lower spacetime dimensions.

As will be shown this is unfortunately not the case. We have succeeded in formulating both the multiplet and the Lagrangian of (old) minimal supergravity regularly in the vierbein, i.e. in a way that they remain well defined, even if $\det e_a{}^m$ vanishes. As this is possible only by introducing a density of negative weight, the resulting Lagrangian is not unique.

It must also be noted that the resulting action is not the supersymmetric extension of the action presented in [1]. This is a consequence of the fact that, as will be shown in Section 3, there is no Palatini formulation of minimal supergravity, a means which is substantial to [1]. In particular we have found no way of describing pure supergravity.

Our approach to supergravity is along the lines of [2], except for the fact that we use the full BRS-transformations of the fields.
In this paper we only outline the strategy leading to our results. The calculational details can be found in [3, 4].

2 The Algebra and the BRS-Transformations

In this section we closely follow [5]. We start investigating the algebra of covariant transformations Δ_N (i.e. transformations that map tensor fields on tensor fields):

$$[\,\Delta_M\,,\,\Delta_N\,]\,T := \big(\Delta_M\Delta_N - (-)^{|M||N|}\Delta_N\Delta_M\big)T = \mathcal{F}_{MN}{}^P\Delta_P T\,, \tag{2.1}$$

where the capital latin indices denote collectively the following operators:

$$\begin{array}{rl} \mathcal{D}_a: & \text{covariant spacetime derivatives,} \\ \mathcal{D}_\alpha,\,\bar{\mathcal{D}}_{\dot\alpha}: & \text{supersymmetry transformations} \\ & \text{(spinor derivatives),} \\ \ell_{[ab]}: & \text{Lorentz spin transformations.} \end{array}$$

The algebra splits into the (super) spacetime transformations ($A \in \{a, \alpha, \dot\alpha\}$) and the Lorentz spin transformations ($[ab]$):

$$\begin{aligned} [\,\mathcal{D}_A\,,\,\mathcal{D}_B\,] &= T_{AB}{}^C\mathcal{D}_C + \tfrac{1}{2}R_{AB}{}^{[ab]}\ell_{[ab]}\,, \\ [\,\ell_{[ab]}\,,\,\mathcal{D}_A\,] &= -G_{[ab]A}{}^B\,\mathcal{D}_B\,, \\ [\,\ell_{ab}\,,\,\ell_{cd}\,] &= \eta_{ad}\,\ell_{bc} - \eta_{ac}\,\ell_{bd} - \eta_{bd}\,\ell_{ac} + \eta_{bc}\,\ell_{ad}\,. \end{aligned} \tag{2.2}$$

The partial derivative acting on tensors defines the connections,

$$\partial_m T = \mathcal{A}_m{}^N\Delta_N T\,, \tag{2.3}$$

which are denoted by

$$\begin{array}{rl} e_m{}^a: & \text{inverse vierbein}^1, \\ \tfrac{1}{2}\psi_m{}^\alpha,\,\tfrac{1}{2}\bar\psi_m{}^{\dot\alpha}: & \text{Rarita-Schwinger field,} \\ -\omega_m{}^{[ab]}: & \text{spin connection.} \end{array}$$

The BRS-transformations are defined similarly, replacing the connections by ghosts

$$sT = C^N\Delta_N T\,. \tag{2.4}$$

By requiring that $[\,s\,,\,\partial_n\,] = s^2 = [\,\partial_n\,,\,\partial_m\,] = 0$ hold on tensor fields, one can deduce the BRS-transformations of the connections and ghosts

$$s\mathcal{A}_m{}^P = \partial_m C^P + C^N\mathcal{A}_m{}^M\mathcal{F}_{MN}{}^P\,, \tag{2.5}$$

[1]Our choice of what is called vierbein and what inverse vierbein is non-standard. It is motivated by the fact that $e_a{}^m$ is the field appearing in typical kinetic terms, e.g. $\eta^{ab}e_a{}^m e_b{}^n\partial_m\phi\,\partial_n\phi^*$. For the same reason the inverse metric was considered fundamental in [1].

$$sC^P = \tfrac{1}{2}(-)^{|N|} C^N C^M \mathcal{F}_{MN}{}^P \,, \tag{2.6}$$

and express some of the structure functions in terms of connections ($\mu \in \{\alpha, \dot\alpha, [ab]\}$):

$$\begin{aligned} \mathcal{F}_{ab}{}^N = -e_a{}^m e_b{}^n \big(&\partial_m \mathcal{A}_n{}^N - \partial_n \mathcal{A}_m{}^N + \mathcal{A}_n{}^\mu \mathcal{A}_m{}^\nu \mathcal{F}_{\mu\nu}{}^N \\ &- \mathcal{A}_n{}^\mu e_m{}^c \mathcal{F}_{\mu c}{}^N + \mathcal{A}_m{}^\mu e_n{}^c \mathcal{F}_{\mu c}{}^N \big) \,. \end{aligned} \tag{2.7}$$

Notice that here and in almost all other intermediate steps we do not question the invertibility of the vierbein. Our final results, however, will remain well defined even if it is noninvertible. These could of course be stated without recourse to the intermediate steps, which are therefore to be regarded as merely motivating.

3 Impossibility of a Palatini formulation

Minimal supergravity is characterized by a set of constraints on the algebra (2.1), and one of these is usually taken to be $T_{ab}{}^c = 0$. Omitting this one (see discussion below), equation (2.7) reads in particular:

$$T_{ab}{}^c = e_a{}^m e_b{}^n (\partial_n e_m{}^c - \partial_m e_n{}^c) + \tfrac{\mathrm{i}}{2}(\psi_a \sigma^c \bar\psi_b - \psi_b \sigma^c \bar\psi_a) + \omega_{ba}{}^c - \omega_{ab}{}^c \,. \tag{3.1}$$

Using the antisymmetry of $\omega_a{}^{bc}$ in its last two indices, one finds

$$\begin{aligned} \omega_{abc} &= \tfrac{1}{2}\big(e_a{}^m e_b{}^n \eta_{cd} + e_c{}^m e_a{}^n \eta_{bd} - e_b{}^m e_c{}^n \eta_{ad}\big)\big(\partial_n e_m{}^d - \partial_m e_n{}^d\big) \\ &\quad + \tfrac{\mathrm{i}}{4}\big(\psi_a \sigma_c \bar\psi_b - \psi_b \sigma_c \bar\psi_a + \psi_c \sigma_b \bar\psi_a - \psi_a \sigma_b \bar\psi_c - \psi_b \sigma_a \bar\psi_c + \psi_c \sigma_a \bar\psi_b\big) \\ &\quad - \tfrac{1}{2}(T_{abc} - T_{acb} - T_{bca}) \\ &=: \Omega_{abc}(e, \psi) - \tfrac{1}{2}(T_{abc} - T_{acb} - T_{bca}) \,. \end{aligned} \tag{3.2}$$

Here one sees that constraining the torsion $T_{ab}{}^c$ to vanish would fix the spin connection in terms of the vierbein, the inverse vierbein and the Rarita-Schwinger field. This function, denoted by $\Omega_a{}^{bc}$, is certainly not regular at vanishing determinant of $e_a{}^m$, hence it must not appear in the transformation laws and the Lagrangian, as these are required to remain well defined if $\det e_a{}^m = 0$. The analogous problem in general relativity was solved in [1] by taking recourse to a Palatini formulation, in which the connection is treated as an independent field whose equations of motion give the above identification.

In this section we show why this is not possible in minimal supergravity, while in the following sections we proceed to show how regular transformations and Lagrangians can be obtained, nonetheless.

It seems that abandoning the constraint $T_{ab}{}^c = 0$ should be sufficient for having a Palatini formulation. To see that it is not, consider the change in the algebra (2.2) induced by

$$\mathcal{D}'_a := \mathcal{D}_a + \tfrac{1}{4}(T_a{}^{bc} - T_a{}^{cb} - T^{bc}{}_a)\, \ell_{bc} \,, \quad \mathcal{D}'_\alpha = \mathcal{D}_\alpha \,, \quad \ell'_{ab} = \ell_{ab} \,. \tag{3.3}$$

It leads to

$$[\, \mathcal{D}'_A \,, \mathcal{D}'_B \,] = T'_{AB}{}^C \mathcal{D}'_C + \tfrac{1}{2} R'_{AB}{}^{ab} \ell_{ab} \,, \tag{3.4}$$

where the primed torsions are given by (the primed curvatures are of no concern to us in the following)

$$T'_{ab}{}^{c} = 0\,, \quad T'_{a\beta}{}^{\gamma} = T_{a\beta}{}^{\gamma} - \tfrac{1}{4}(T_{abc} - T_{acb} - T_{bca})\sigma^{bc}{}_{\beta}{}^{\gamma}\,. \tag{3.5}$$

No other torsions are changed, in particular

$$T'_{ab}{}^{\gamma} = T_{ab}{}^{\gamma}\,. \tag{3.6}$$

Expressing (2.3) and (2.4) in the new basis, one also obtains a change in the ghosts and connections:

$$\begin{aligned} sT = C^N \Delta_N T &= C'^N \Delta'_N T \\ \Longrightarrow C'^A = C^A\,, \quad C'^{[ab]} &= C^{[ab]} + \tfrac{1}{2}C^a(T_a{}^{bc} - T_a{}^{cb} - T^{bc}{}_a) \end{aligned} \tag{3.7}$$

$$\begin{aligned} \partial_m T = \mathcal{A}_m{}^N \Delta_N T &= \mathcal{A}'_m{}^N \Delta'_N T \\ \Longrightarrow \mathcal{A}'_m{}^A = \mathcal{A}_m{}^A\,, \quad \omega'_m{}^{[bc]} &= \omega_m{}^{[bc]} + \tfrac{1}{2}e_m{}^a(T_a{}^{bc} - T_a{}^{cb} - T^{bc}{}_a)\,. \end{aligned} \tag{3.8}$$

One sees that this change of basis is equivalent to imposing the constraint $T_{ab}{}^c = 0$. All this is not unlike the case of general relativity. What has been said up to now is hardly more than an elaborate way of stating that two connections differ by a tensor.

The distinction comes when one starts to write down Lagrangians. In general relativity one can use the Ricci scalar expressed in terms of the independent or dependent spin connection according to the algebra used. Supergravity, however, is more restrictive. The Ricci scalar enters the action through the $\mathcal{D}^2 M$ term in the Lagrangian (compare with (5.4): $\mathcal{D}^2$ comes from the $\hat{F}$ term and M, the auxiliary scalar in the supergravity multiplet, from the chiral projector). From the Bianchi identities (we use the conventions of [3, 4]) one knows:

$$\mathcal{D}^\alpha \mathcal{D}_\alpha M = \tfrac{8}{3}\sigma^{ab}{}_\gamma{}^\alpha \mathcal{D}_\alpha T_{ab}{}^\gamma\,. \tag{3.9}$$

But the derivatives as well as the torsions which appear in this equation are left invariant by the above change of basis (3.3), (3.6), so that one also has

$$\mathcal{D}^\alpha \mathcal{D}_\alpha M = \tfrac{8}{3}\sigma^{ab}{}_\gamma{}^\alpha \mathcal{D}'_\alpha T'_{ab}{}^\gamma\,. \tag{3.10}$$

But this means that this part of the Lagrangian is not altered at all by the change of basis and therefore a Palatini formulation does not exist in minimal supergravity. One can understand this result in the following way:

A crucial difference between general relativity and supergravity is the fact that in the latter theory all curvatures are determined by the torsions [6]. But this means that changing the constraints on the torsions will also effect the curvatures and in particular the Ricci scalar. Indeed, one finds that abandoning $T_{ab}{}^c = 0$ allows to express the Ricci scalar in terms of an independent spin connection, but on the other hand one is forced to add additional torsion terms to the action. If now (3.1) is inserted for $T_{ab}{}^c$, all terms containing the independent spin connection (ω) cancel and one is left with the Ricci scalar expressed in the dependent spin connection (Ω).

In formulae (denoting the torsion terms by $f(T)$):

$$R(\Omega) \stackrel{T\neq 0}{\Longrightarrow} R(\omega) + f(T) = R(\omega) + f(T(\omega,\,\Omega)) = R(\Omega)\,. \tag{3.11}$$

But this is just the above statement: Abandoning the constraint $T_{ab}{}^c = 0$ does not alter the Ricci scalar part of the supergravity Lagrangian. Explicit calculation shows that the entire Lagrangian is unaffected by this change of basis of the algebra.
One also sees that it is possible to express the transformation laws of the fields entirely in terms of the dependent spin connection as the change of basis leaves the BRS-operator invariant.

4 Regular Transformations

Looking at the explicit form of the BRS-transformations (see e.g. [2]) one sees that these are not regular expressions in $e_a{}^m$, due to covariant derivatives appearing e.g. in the transformation law of the Rarita-Schwinger field. These contain the spin connection, which, as was demonstrated, is not regular. However, one also sees that terms containing the spin connection are the only singular terms, if one uses $\{e_a{}^m,\, \mathcal{A}_a{}^\mu = e_a{}^m \mathcal{A}_m{}^\mu\}$ instead of the $\mathcal{A}_m{}^N$ as fundamental fields.
To obtain a regular multiplet we introduce a vierbein density as a fundamental field:

$$\begin{aligned} E_a{}^m &:= e^{-q} e_a{}^m\,, \quad & E_m{}^a &= e^q e_m{}^a\,, \\ e_a{}^m &= E^\kappa E_a{}^m\,, \quad & e_m{}^a &= E^{-\kappa} E_m{}^a\,, \end{aligned} \tag{4.1}$$

where $e = \det e_a{}^m$, $E = \det E_a{}^m$, $\kappa = q/(1-4q)$ and q is a real number to be specified later. The above expressions can obviously not be used to define the new fields at vanishing e. They allow, however, to calculate the BRS-transformations of these fields which remain well defined even at $e = 0$ and serve as the proper definition of $E_a{}^m$. (4.1) then shows how to regain the old vierbein at nonvanishing determinant.
We can now examine the singular terms contained in the spin connection. These are all of the form

$$e_b{}^m e_a{}^n \partial_n e_m{}^c = -e_m{}^c e_a{}^n \partial_n e_b{}^m \tag{4.2}$$

and read, expressed in the new fields:

$$E_m{}^c E_a{}^n \partial_n (E^\kappa E_b{}^m) = E^\kappa E_m{}^c E_a{}^n \partial_n E_b{}^m + \kappa \delta_b^c E_a{}^n E^\kappa E_m{}^d \partial_n E_d{}^m\,. \tag{4.3}$$

The inverse of a matrix is given by a polynomial expression in its matrix elements (called the minor) devided by its determinant. This means that $E E_m{}^a$ is a regular expression in $E_a{}^m$ even though $E_m{}^a$ is not. In particular (4.3) is regular in the $E_a{}^m$ if only κ is chosen to be greater than or equal to one. It is polynomial if κ is a natural number.
Thus it is immediately obvious that by densitizing the vierbein all singularities can be removed from the transformation laws of the ghosts and the connections and only

the transformation of the vierbein density itself remains to be checked. One finds

$$
\begin{aligned}
sE_a{}^m = {} & \frac{\kappa}{1+4\kappa}\big(E^\kappa E_b{}^n\partial_n C^b + C^N\mathcal{A}_b{}^M\mathcal{F}_{MN}{}^b\big)E_a{}^m \\
& - E^\kappa E_b{}^m E_a{}^n\partial_n C^b - E_b{}^m C^N\mathcal{A}_a{}^M\mathcal{F}_{MN}{}^b\,,
\end{aligned}
\tag{4.4}
$$

which again is regular.
The restriction $\kappa \geq 1$ translates to the requirement that q be from the half-open interval $[\frac{1}{5}\,,\,\frac{1}{4})$. κ is a natural number if q has the form $q = n/(1+4n)$ with n natural.
If one now calculates the transformation of E, one finds

$$
\begin{aligned}
sE &= (4q-1)\big[\partial_n(E^\kappa E_a{}^n C^a) + \mathrm{i}\big((C - \tfrac{1}{2}C^a\psi_a)\sigma^b\bar\psi_b + \psi_b\sigma^b(\bar C - \tfrac{1}{2}C^a\bar\psi_a)\big)\big]E \\
&\quad + E^\kappa E_a{}^n C^a\partial_n E \\
&= (4q-1)\big[\partial_n\hat C^n + \mathrm{i}(\hat C\sigma^b\bar\psi_b + \psi_b\sigma^b\hat{\bar C})\big]E + \hat C^n\partial_n E\,,
\end{aligned}
\tag{4.5}
$$

where

$$
\hat C^n := E^\kappa E_a{}^n C^a\,,\quad \hat C^\alpha := C^\alpha - \tfrac{1}{2}C^a\psi_a{}^\alpha\,. \tag{4.6}
$$

Recognizing that the first term is the supersymmetric generalization of the characteristic term in the transformations of densities, one sees that E transforms as a density of weight $4q-1$, which is negative in the allowed range of q.
The results of this section show that one can obtain a regular second order formulation of general relativity if one uses vierbein densities as fundamental fields. One cannot achieve this in a metric/affine connection formulation by introducing (inverse) metric densities.

5 The Lagrangian

In minimal supergravity one constructs Lagrangians from chiral fields. This cannot at once be generalized to our case. The problem is that a Lagrangian has to transform as a density of weight one, which is usually achieved by choosing a scalar times the determinant of the inverse vierbein ($\det e_m{}^a$) as the Lagrangian. This, of course, is not possible here, as we want to investigate theories in which $e_m{}^a$ need not exist. The remedy is to investigate not ordinary chiral fields, but chiral densities, i.e. to construct a Lagrangian from fields that themselves contribute weight, so that no explicit determinants are needed. As then the weight of the Lagrangian comes from matter fields we will however not be able to describe pure supergravity.
$(\hat\phi\,,\hat\chi_\alpha\,,\hat F)$ being the components of a chiral multiplet, we define

$$
\phi := e^{-p}\hat\phi\,,\quad \chi_\alpha := e^{-p}\hat\chi_\alpha\,,\quad F := e^{-p}\hat F\,,\qquad e = E^{1/(1-4q)}\,(=\det e_a{}^m)\,. \tag{5.1}
$$

"Defining" has, as in (4.1), to be regarded as tongue-in-cheek. (5.1) motivates the (regular) transformations of these fields which in turn are used to define them properly. These transformations will however not be explicitely stated here as they

would introduce technicalities not necessary to understand the following. It shall suffice to mention the instances in which chiral densities require treatment different from chiral fields.

Chiral fields are characterized by the equation $\bar{\mathcal{D}}_{\dot\alpha}\hat\phi = 0$, which is equivalent to the requirement that their transformation must not contain a ghost $\bar{C}^{\dot\alpha}$. Looking at the characteristic term for the transformations of densities (4.1), one sees that this condition has to be altered. It turns out to be sufficient to subtract the extra term. Introducing the new operators

$$\delta_\alpha := \Big[\frac{\partial}{\partial \hat{C}^\alpha}\,,\,s\Big] - \mathrm{i}(\sigma^a\bar\psi_a)_\alpha\,\mathcal{W}\,,\quad \bar\delta_{\dot\alpha} := -\Big[\frac{\partial}{\partial \hat{\bar{C}}^{\dot\alpha}}\,,\,s\Big] + \mathrm{i}(\psi_a\sigma^a)_{\dot\alpha}\,\mathcal{W}\,, \tag{5.2}$$

where $\mathcal{W}$ assigns to a field its weight (e.g. $\mathcal{W}(\phi) = p\phi$), one can write the defining equation for chiral densities as $\bar\delta_{\dot\alpha}\phi = 0$. Also the equations which give the higher components of chiral fields can be generalized to chiral densities:

$$\chi_\alpha = \delta_\alpha\phi\,,\quad F = -\tfrac{1}{4}\delta^2\phi\,. \tag{5.3}$$

The corresponding equations hold for antichiral densities[2].

As δ_α and $\bar\delta_{\dot\alpha}$ satisfy the (graded) Leibniz rule a product of chiral densities is again a chiral density, the weight being the sum of the weights of its factors.

The chiral projector known from minimal supergravity generalises to chiral densities, i.e. the operator $(\bar\delta^2 - 2M)$ acting on a product of chiral and antichiral densities gives a chiral density. The weight of this density is again the sum of the weights of the factors. The proof is straightforward, but will be omitted here as we have not discussed the transformation laws of the auxiliary fields.

With these results it is now easy to construct Lagrangians. From minimal supergravity it is known that

$$e^{-1}\big(\hat F + \tfrac{\mathrm{i}}{2}\bar\psi_a\bar\sigma^a\hat\chi + \tfrac{3}{2}M^*\hat\phi - \bar\psi_a\bar\sigma^{ab}\bar\psi_b\,\hat\phi\big)\,,\quad e = E^{1/(1-4q)}\,(=\det e_a{}^m) \tag{5.4}$$

transforms into a total derivative. But by definition

$$\begin{aligned} &F + \tfrac{\mathrm{i}}{2}\bar\psi_a\bar\sigma^a\chi + \tfrac{3}{2}M^*\phi - \bar\psi_a\bar\sigma^{ab}\bar\psi_b\,\phi \\ &= \big(-\tfrac{1}{4}\delta^2 + \tfrac{\mathrm{i}}{2}\bar\psi_a\bar\sigma^a\delta + \tfrac{3}{2}M^* - \bar\psi_a\bar\sigma^{ab}\bar\psi_b\big)\phi \end{aligned} \tag{5.5}$$

has the same transformation law and therefore gives a Lagrangian, if the chiral density ϕ has weight one, without an inverse vierbein appearing explicitly. Since we know how to construct chiral densities from chiral and antichiral densities we can immediately give the formula for the most general Lagrangian we are able to build:

$$\begin{aligned} \mathcal{L} = &\big(-\tfrac{1}{4}\delta^2 + \tfrac{\mathrm{i}}{2}\bar\psi_a\bar\sigma^a\delta + \tfrac{3}{2}M^* - \bar\psi_a\bar\sigma^{ab}\bar\psi_b\big)\big[(\bar\delta^2 - 2M)\,K(\phi,\phi^*) + g(\phi)\big] \\ &+ \text{h.c.}\,. \end{aligned} \tag{5.6}$$

[2]The operators δ_α and $\bar\delta_{\dot\alpha}$ should not be regarded as generalising the spinor derivatives to act on tensor densities. While they reduce to $\mathcal{D}_\alpha$ and $\bar{\mathcal{D}}_{\dot\alpha}$ when acting on tensors, they do not satisfy the algebra (2.1) on tensor densities.

Here K is a polynonial of chiral and antichiral densities, while g is a polynonial of chiral densities only. In both cases each individual monomial must be a density of weight one.
One has to remember, however, that the theory contains a chiral field of negative weight, namely E, (4.1). This means that one can take an arbitrary polynomial in elementary chiral densities and multiply its terms with appropriate powers of E to obtain again a chiral density of weight one. This in turn demonstrates that one does not obtain a unique action from symmetry and regularity alone. Indeed, one sees that to every action of minimal supergravity coupled to chiral matter there exists a corresponding regular action.

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